Algebraic Topology and Lefschetz Fixed Point Theorem

given a continuous map $f: X \to X$ from a topological space to itself, theorem will say if it has fixed points and how many

$\hookrightarrow f(x) = x$

$$\Lambda(f) = \sum_n (-1)^n \underline{tr(f_*: H_n(X) \to H_n(X))}$$

- alternating sum of the trace of f_* on homology groups

Homology

- concept in algebraic topology used to study topological spaces by breaking them down to loops, cavities?
- about "holes" different dimensions

H_0 : connected components

H_1 : loops

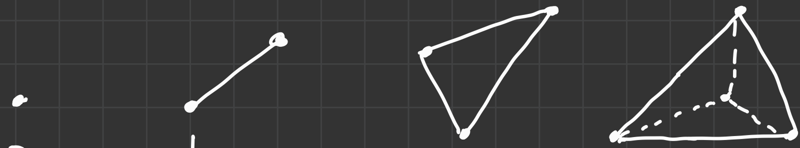
H_2 : cavities

$\vdots$

H_n : higher dimension holes

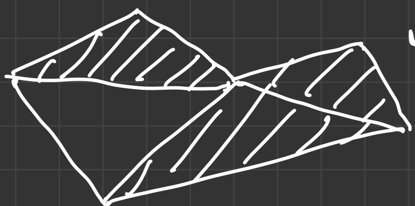
torus

Simplicial Complex



0   1   2 (triangle)   3 (tetrahedron)   ? ⋮ n (n-simplex)

boundary of a q-simplex is made of q-1 simplexes

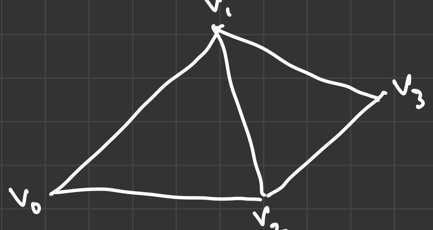


also simplicial complex

def a simplicial complex K consists of a set $\{v\}$ of vertices and a set $\{s\}$ of finite non-empty subsets of $\{v\}$ called simplexes s.t.

a) any set containing exactly one vertex is a simplex

b) any nonempty subset of a simplex is a simplex



simplicial complex: $\{v_0, v_1, v_2, v_3\}$

0-simplices $\{v_0\}, \{v_1\}, \ldots$

1-simplices $\{v_0, v_1\}, \ldots$ edges

2-simplices $\{v_0, v_1, v_2\}, \ldots$ cycles

notes:

- simplex s with q+1 vertices is called a q-simplex
  - dimension of s is q

- if $s' \subset s$, s' is called a face of s
  - or proper face if $s' \neq s$
  - if s' is a p-simplex, also called a p-face

$\Rightarrow$ any simplex is determined by its 0-faces or 0-simplices (vertices)

from a, b

- dimension of a simplicial complex K is
  - n if K contains an n-simplex, but no (n+1) simplex
  - -1 if K is empty
  - ∞ if K contains n-simplices $\forall n \geq 0$

examples

a) empty set of simplexes is a simplicial complex $\emptyset$ (dim -1)

b) extending previous example $\{v_0, v_1, \dots, v_n\}$ $n \in \mathbb{Z}^+$

↳ infinite simplicial complex with dim $K = \infty$

c) simplicial complex with set of vertices $\{v\} = \mathbb{Z}$ and simplexes $\{\{n\} \mid n \in \mathbb{Z}\} \cup \{\{n, n+1\} \mid n \in \mathbb{Z}\}$

↳ infinite simplicial complex with dim $K = 1$

<u>def</u> a <u>simplicial map</u> $\phi: K_1 \to K_2$ is a function ϕ from vertices of K_1 to vertices of K_2 s.t.

for any simplex $s \in K_1$, $\phi(s)$ is a simplex of K_2

"simplicial approximation"

method of approximating a continuous map f between topological spaces by triangulation into simplicial complexes

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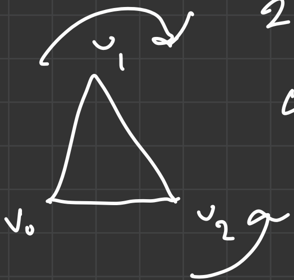

2-simplex $\{v_0, v_1, v_2\}$

can also write $\{v_0, v_2, v_1\}$) cycles

$\{v_0, v_1\}$ $\{v_1, v_2\}$ $\{v_2, v_0\}$

$\{v_1, v_0\}$ $\{v_2, v_1\}$ $\{v_0, v_2\}$) edges

cycles - linear combination of edges

boundaries - cycles that are filled

q-simplex s^q can be $+s^q$ or $-s^q$

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let S^q be the set of all q-simplexes of K where $0 \leq q \leq \dim K$

considering $+s^q$ and $-s^q$ as different, set of all q-simplexes is denoted $\bar{S}^q$

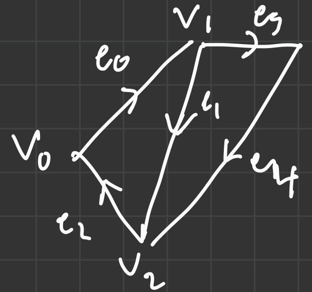
$|\bar{S}^q| = 2|S^q|$, $|S^0| = |\bar{S}^0|$

def consider a finite simplicial complex K with simplexes
$\tilde{S}^q$, $0 \le q \le \dim(K)$

a map $f: \tilde{S}^q \to \mathbb{Z}$ is called a q-chain, if $f(-s^q) = -f(s^q)$
$\forall\ s^q \in \tilde{S}^q$

define chain group $(C_q(K), +)$ q-chains under addition

$$C_q(K) = \bigoplus \mathbb{Z} s^q$$



$$C_0 = \left\{ \sum_{i=0}^{4} a_i v_i \mid a_i \in \mathbb{Z} \right\}$$
linear combinations

<u>boundary map</u>

(for particular n

let $B_n(K)$ be the set of n-simplexes in K
$B_n(K)$ is a basis for the free abelian group $C_n(K)$
(group of n-chains in K)

define function
$$\partial_n' : B_n(K) \to C_{n-1}(K)$$

for any n-simplex s or $\sigma = [v_1, v_1, \dots v_n]$

$$\partial_n'(\sigma) = \partial_n'([v_0, \dots v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

alternating sum of n-1 simplexes where ith term missing ith vertex

<u>def</u> boundary map of dimension n is the unique homomorphism $\partial_n : C_n(K) \to C_{n-1}(K)$ given by extending ∂_n' by linearity

form a sequence

$$\xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

boundary maps well defined $\partial_n(-\sigma) = -\partial_n(\sigma)$

boundary of a boundary is always 0

Thm for simplicial complex K with $\sigma \in K$ an n-simplex $(\partial_n \circ \partial_{n+1})(\sigma) = 0$

$$(\partial_{n-1} \circ \partial_n)([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i \partial_{n-1}[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

$$= \sum_{j<i} (-1)^i (-1)^j [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n]$$

$$= 0$$

(kernel subgroup of domain that maps to 0
image basically range, subgroup of codomain)

Thm for simplicial complex K

$$\operatorname{im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n \qquad n \in \mathbb{Z}^+$$

for any $\beta \in \operatorname{im} \partial_{n+1}$ $\quad \beta = \partial_{n+1}(\sigma)$ for some chain $\sigma \in C_{n+1}$ $\quad \partial_n(\beta) = \partial_n(\partial_{n+1}(\sigma)) = 0$

so $\beta \in \ker \partial_n \Rightarrow \partial_{n+1} \subseteq \ker \partial_n$

because these are abelian groups
$\Rightarrow$ they are normal subgroups
since $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$
$\operatorname{im} \partial_{n+1}$ is a normal subgroup of $\ker \partial_n$

can form quotient group $\dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$

<u>def</u> homology group

$$H_i(C) := \frac{\ker(\partial_i)}{\operatorname{im}(\partial_{i+1})}$$

<u>References:</u>

Algebraic Topology by Edwin H. Spanier

An Overview and Proof of The Lefschetz Fixed Point Theorem by Edgar Lin

Eric Shapiro notes on Simplicial Complexes and Boundary Maps